Engaging Students in Proof and Reasoning in High School Non-Geometry Courses

GRADES 9-12

Participant Workbook

Revised April 2016
What is proof?

Write your definition here and we will revisit and possibly revise it at different times during the professional development.

Reflect: How do your students determine whether a mathematical statement is true or false?
Purposeful Pedagogy and Discourse Instructional Model: Student Thinking Matters Most

In studying the Common Core State Standards for Mathematics (CCSSM), and in particular the Standards for Mathematical Practice (SMP), it becomes clear that what we do in the classroom will change both from the perspective of the teacher and the student. The teacher will need a deep and connected understanding of the mathematics content and, during instruction, will need to provide experiences that allow the students to construct meaning for themselves through carefully crafted tasks and conversations. Students will need to reason, communicate, generalize and challenge the mathematical thinking of themselves and others. Student thinking matters most.

The purposeful pedagogy and discourse instructional model that we are using in the Arkansas CCSS Mathematics Professional Development Project, is based on the research of four sets of researchers:

• Jacobs, Lamb, and Philipp on professional noticing and professional responding;
• Smith, Stein, Hughes, and Engle on orchestrating productive mathematical discussions;
• Ball, Hill, and Thames on types of teacher mathematical knowledge;
• Levi and Behrend (Teacher Development Group) on Purposeful Pedagogy Model for Cognitively Guided Instruction.

This model is intended to support teachers to deliver strong mathematical content using critical best classroom practices as well as to develop a learning environment where their students regularly use the 8 Standards for Mathematical Practice.

Assessing Students, Professional Noticing, and Teacher Mathematical Knowledge

At the core of our model is assessing students (TDG-CGI model), which refers to taking a close look at student understanding. While assessing students, we apply the concept of professional noticing (Jacobs et al.).

Professional noticing is comprised of 3 teacher skills:

• Attending to children’s strategies,
• Interpreting children’s understanding, and
• Deciding on how to respond on the basis of children’s understanding.

In order to assess a students’ understanding, we must look at the details of their thinking (what did they do) and then mathematically interpret these details. While this may seem trivial, students’ strategies are complex and many deep mathematical operations and properties are embedded implicitly in their work. It takes time to identify the important details in students’ thinking and then mathematically interpret the relationships and properties of operations that are embedded. The ability to notice will help the teacher identify the mathematics available for exploration during the lesson(s) to follow. Since student thinking matters most, in the Arkansas professional development courses the beginning of most classes will involve just making sense of and deepening our understanding of the details of students’ strategies and the mathematical ideas embedded in their strategies.

The deeper and more connected a teacher’s mathematical knowledge is, the easier it is to see and interpret the details of student thinking. Teaching mathematics requires a variety of types of knowledge as shown in Figure 1.

Written by Linda Jaslow in collaboration with Aimee L. Evans
One type of teacher mathematical knowledge is *specialized content knowledge* — the mathematics behind the mathematics. For example, it is not enough to know we can divide fractions by inverting the second fraction and multiplying. A teacher must understand the mathematics that allows that strategy to work. Teachers must also understand how children will approach various problems, how their thinking develops, and how students’ thinking is different than adults’ thinking. This knowledge is called *knowledge of content and students*. All of this comes together to create the critical part of professional noticing, identifying the details of children’s thinking and mathematically interpreting the details, which allows us to assess students’ thinking, which of course matters above all else.

**Exercising Professional Noticing**

A fourth grade student solved the following problem: Kathy is making ____ cupcakes. She put ____ cups of frosting on each cupcake. How many cans of frosting will she need to make her cupcakes?

Two sets of numbers: (36, ¼) (36, ¾)

**Figure 2: Student work on 36 x 1/4**

What did this student do? What big mathematical ideas are embedded in her strategy? Take a few minutes to follow her trail of thinking. How would you mathematically notate her reasoning? See Figure 2.

What is it that teachers have to know to be able to understand the mathematics of this students thinking? It is not enough to know the properties of operations, teachers need to have a deeper understanding of
these properties and be able to interpret this important mathematics embedded in student informal strategies.

To solve the problem using the first set of numbers, the student first transformed the problem with commutative property $36 \times \frac{1}{4} = \frac{1}{4} \times 36$. She then solved by first finding that $\frac{1}{2}$ of $36 = 18$, and then finding that $\frac{1}{2} \times 18 = 9$.

What mathematics allows for this sequence of thinking?

$$
36 \times \frac{1}{4} = \frac{1}{4} \times 36 \quad \text{Commutative property}
$$

$$
\frac{1}{4} \times 36 = \left(\frac{1}{2} \times \frac{1}{2}\right) \times 36 \quad \text{Decomposing}
$$

$$
\left(\frac{1}{2} \times \frac{1}{2}\right) \times 36 = \frac{1}{2} \times \left(\frac{1}{2} \times 36\right) \quad \text{Associative Property}
$$

The student then used the relationship between $\frac{1}{4}$ and $\frac{3}{4}$ to solve the problem with the other set of numbers.

**Professional Responding, Purposeful Pedagogy, and Orchestrating Classroom Discourse**

Critical instructional decisions are based on the mathematical interpretation of students understanding. With specialized content knowledge and knowledge of content and students in place, we are ready to focus on our mathematical practice. The Purposeful Pedagogy Model (TDG; Cognitively Guided Instruction) and Orchestrating Classroom Discourse (Stein et al.) come together to give us a vision of such practice centered around the all important student thinking.

The Purposeful Pedagogy Model has three components: assess students, set a learning goal, and design instruction.

Elements for the design of the instruction are defined by the Orchestrating Classroom Discourse research.

Orchestrating Classroom Discourse outlines 5 practices for doing so:

1. **Anticipating** likely student responses to cognitively demanding mathematical tasks;
2. **Monitoring** students’ responses to the tasks during the explore phase;
3. **Selecting** particular students to present their mathematical responses during the discuss-and-summarize phase;
4. **Purposefully sequencing** the student responses that will be displayed;
5. Helping the class make mathematical connections between different students’ responses and between students’ responses and the key ideas.

We will use the details of student understanding to set learning goals for our students, design instruction, and orchestrate classroom discourse. In doing so, we are engaging in the comprehensive practice of professional responding. This is best understood by taking a look at a classroom vignette from Kindergarten.

The students in this class have been solving problems that begin with 10 and add some more. The teacher has elected to present this problem by beginning with an amount other than 10 and then adding on 10 to see how students will respond. Before reading the classroom exchange and the teacher’s

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professional responding, look at the student work from a kindergarten class for the following problem and answer these questions for yourself: Zayeqwain had 6 pennies. He gets 10 more. How many pennies does he have now?

• What did the students do?
• What is the mathematics embedded in their strategies?
• How are the strategies alike and different?
• Why do you think the teacher would have selected these two students to share?
• What conversation do you think the teacher would like to have?

Classroom Vignette
The classroom teacher, Mrs. J asked the two students to share their solutions with the class and then engaged the class in a discussion around their strategies.

Pretty: There are 10 [pennies], (then she counted on) 11, 12, 13, 14, 15, 16.
Moniqua: There are 6, (then counted) 7, 8, 9, 10, 11, 12, 13, 14, 15, 16. And look I came up with 2 number sentences (excitedly)
6 + 10 = 16 and 10 + 6 = 16. See I can do it two ways!
Mrs. J.: Look at these two strategies. Are they alike or different?
Sandia: They are alike. They both counted up.
Mrs. J.: I can see that they both used a counting up strategy. What do the rest of you think?
Theo: No, they are not alike. They started counting from a different number. Moniqua started counting from 6 and Pretty started counting from 10.
Mrs. J.: (pointing at Moniqua’s number sentences) So, which one of Moniqua’s number sentences go with the problem?
Class: 6 + 10 = 16.
Mrs. J.: Why?
Claudette: Because Zayeqwain has 6 pennies and then he gets 10 more.
Mrs. J.: Do any of these number sentences represent Pretty’s strategy?
Claudette: 10 + 6 = 16.

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Mrs. J.: Why?
Maria: Because she started with 10 first and then added her 6 seconds.
Mrs. J.: Is that okay to do?
Class: Yes. No. (mixed answers)
Mrs. J.: Will they both get the same answer?
Cecilia: I just counted it on my fingers. They are both 16. (The class is surprised.)
Mrs. J.: Really? Do you think this was just an accident, or do you think this will always happen?
Cecilia: It won’t always happen, just on this problem.

Mrs. J decided to stop after this exchange and let her students’ ideas percolate. About a week later, when she posed a similar problem (4 + 10), five additional students switched the order of the numbers to solve the problem and counted on from 10 instead of 4, utilizing the commutative property of addition. After further discussion, many of the students were beginning to think that this might be something that would always work.

How is this episode related to the purposeful pedagogy and discourse instructional model? The teacher posed a problem to her class and allowed the students to solve the problem the way that made sense to them. She identified student work that had the potential to help her students discover and make sense of an important mathematics concept. Specifically, when Pretty counted on from the larger number, the teacher understood Pretty’s strategy was based on the commutative property. The teacher also noticed Moniqua’s number sentences, 6 + 10 = 16 and 10 + 6 = 16. Based on her analysis and observation, she made an instructional decision to use this as an opportunity to have class discussion about the commutative property and how number sentences relate to the structure of the problem. As opposed to telling the students that this was a “turn around fact” or to “just count on from the larger number,” she put the students in the position to consider these complex ideas for themselves by facilitating the dialogue to help them make meaning connected to their existing thinking.

While this type of exchange requires the classroom teacher to think very purposefully about instructional decisions and to think deeply about the mathematics embedded in students’ solutions, the effort is worthwhile. The evidence comes from Cognitively Guided Instruction, an instructional model that emphasizes these very practices. Visits to CGI classrooms in Arkansas will reveal that children are thinking more deeply and flexibly about mathematics. They are not simply solving problems that have no meaning to them; they are becoming young mathematicians capable of explaining their thinking, which matters most, and grappling with and making sense of the complexity of the mathematics.

How do we now take the information we have about students’ thinking and professionally respond in a way that is based on students’ understanding and designed to facilitate children’s thinking along a learning trajectory? We must select or design appropriate mathematical tasks or problems.

Mathematical tasks should be selected that will facilitate children’s development. Once we have identified the task, we should consider the following questions:
- What do we anticipate students will do with the task?
- Will this task provide the experiences needed to further students’ development?
- Which of the strategies we expect are likely to help the most in making sense of the mathematics in the goals we have set for them?

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The next stage is to pose the task or problem and allow the students to solve the problem in a way that makes the most sense to them. Our job is to monitor students to identify what students are doing, guide them as they work, and decide which students’ papers should be shared.

**Back to Teacher Mathematical Knowledge**

Once we have identified the best student strategies to meet the learning goals, we need to decide in which order to share students’ strategies and what mathematical connections should be the focus of the classroom discussion. Again, the teacher’s mathematical knowledge, specifically her *knowledge of content and teaching* (Hill & Ball, Figure 1), will be critical in making decisions by being able to envision how the mathematics available through the students’ strategies connect to one another and to the mathematics concepts that are desired.

At this juncture, the teacher’s knowledge of the mathematics meets the need to design or plan the discourse to take students deeper into the mathematics. This involves both the sequencing of the presentation and also the selection and phrasing of the questions posed during the discourse. There are likely multiple productive paths, but there are certainly some unproductive or problematic paths as well, and the teacher will need to choose well. Student thinking matters most.

**Seeing It All Together**

The research of these four sets of researchers come together to create the instructional model that we are using in the courses for the Arkansas CCSS Mathematics Professional Development Project. While this model, being a blend of the work of so many different projects, may seem complex at first, it is perhaps more straight forward when viewed using the graphic organizer below. The key ideas that hold the model together are the importance of noticing the details of student thinking, interpreting those details, and using that information to design instruction comprised of discourse around student strategies aimed at a specific mathematical goal. In other words, what details do we see in our children’s work, how do we interpret their thinking, and where mathematically do we go from there? In maintaining this focus throughout the professional development courses, it is our hope to support teachers in their journey toward achieving mathematical proficiency for their students as described in the CCSSM. And always remember, student thinking matters most.

REFERENCES:


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Arkansas CCSSM Professional Development Purposed Pedagogy and Discourse Instructional Model

Key:
- Orchestrating Classroom Discourse
- Purposed Pedagogy Model

1. Write or select a problem or task.
2. Anticipate what students will do that might be productive.
3. Pose problems and monitor students as they solve.
4. Select student work to share that would be productive.
5. Sequence the papers to share to help students make connections.
6. Compare and contrast strategies and make mathematical connections (discourse).

Professional Noticing:
- Identify children’s thinking and interactions during the lesson.
- Identify details of children’s thinking.
- Identify knowledge of content.
- Identify specialized content knowledge. (High & Ball)
- Identify how students’ ideas change and develop over time.

Professional Noticing:
- Identifying details of children’s thinking.
- Identifying knowledge of content.
- Identifying specialized content knowledge. (High & Ball)
- Identifying how students’ ideas change and develop over time.

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Standards that Indicate Proof, Explain, or Understand:

- **MP 3** – Construct viable arguments and critique the reasoning of others.
- **Other Mathematical Practices** are related to proof (will be discussed in PD).
- **HSN.RN.B.3** – Explain why the sum/difference or product/quotient of two rational numbers is rational; • The sum/difference of a rational number and an irrational number is irrational; • The product/quotient of a nonzero rational number and an irrational number is irrational; and • The product/quotient of two nonzero rationals is a nonzero rational.
- **HSN.CN.A.2** – Use the relation \( i^2 = -1 \) and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
- **HSN.CN.C.7** – Solve quadratic equations with real coefficients that have real or complex solutions.
- **HSA.SSE.B.4** – Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*
- **HSA.APR.C.4** – Prove polynomial identities and use them to describe numerical relationships. *Examples of Polynomial Identities may include but are not limited to the following:* • \((a + b)^2 = a^2 + 2ab + b^2\) (Algebra 1) • \(a^2 - b^2 = (a + b)(a - b)\) (Algebra 1) • \((x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2\) can be used to generate Pythagorean triples (Algebra 2).
- **HSA.RELA.1** – Assuming that equations have a solution, construct a solution and justify the reasoning used. Note: Students are not required to use only one procedure to solve problems nor are they required to show each step of the process. Students should be able to justify their solution in their own words.
- **HSA.RELD.11** – Explain why the x-coordinates of the points where the graphs of the equations \( y = f(x) \) and \( y = g(x) \) intersect are the solutions of the equation \( f(x) = g(x) \) [see actual standard for many specifications and notes].
- **HSF.TF.A.2** – Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.
- **HSF.TF.C.8** – (+) Develop the Pythagorean identity \( \sin^2(\theta) + \cos^2(\theta) = 1 \). Given \( \sin(\theta) \), \( \cos(\theta) \), or \( \tan(\theta) \) and the quadrant of the angle, use the Pythagorean identity to find the remaining trigonometric functions.
- **HSG.CO.A.4** – Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
- **HSG.CO.B.8** – Investigate congruence in terms of rigid motion to develop the criteria for triangle congruence (ASA, SAS, SSS, and HL). *Note: The emphasis in this standard should be placed on investigation.*
- **HSG.CO.C.9** – Apply and prove theorems about lines and angles. *Theorems include but are not limited to:* vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a
perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.

- **HSG.CO.C.10** – **Apply and prove theorems about triangles.** Theorems include but are not limited to: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point. Note: Proofs are not an isolated topic and therefore should be integrated throughout the course.

- **HSG.CO.C.11** – **Apply and prove theorems about quadrilaterals.** Theorems include but are not limited to relationships among the sides, angles and diagonals of quadrilaterals and the following theorems concerning parallelograms: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely rectangles are parallelograms with congruent diagonals. Note: Proofs are not an isolated topic and should be integrated throughout the course.

- **HSG.CO.D.12** – **Make formal geometric constructions** with a variety of tools and methods ~(compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). **Constructions may include but are not limited to:** copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line. Note: Constructions are not an isolated topic and therefore should be integrated throughout the course.

- **HSG.SRT.A.2** – Given two figures • Use the definition of similarity in terms of similarity transformations to determine if they are similar • Explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

- **HSG.SRT.A.3** – Use the properties of similarity transformations to establish the AA, SAS~, SSS~ criteria for two triangles to be similar.

- **HSG.SRT.B.4** – Use triangle similarity to apply and prove theorems about triangles. **Theorems include but are not limited to:** a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity. [Standards contain an example with a diagram]

- **HSG.SRT.B.5** – Use congruence (SSS, SAS, ASA, AAS, and HL) and similarity (AA, SAS~, SSS~) criteria for triangles to solve problems. Use congruence and similarity criteria to prove relationships in geometric figures.

- **HSG.SRT.C.7** – **Explain** and use the relationship between the sine and cosine of complementary angles.

- **HSG.C.A.1** – Prove that all circles are similar. [example in standards]

- **HSG.C.A.3** – • Construct the inscribed and circumscribed circles of a triangle. •Prove properties of angles for a quadrilateral inscribed in a circle.

- **HSG.C.B.5** – • Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius • Derive and use the formula for the area of a sector
• Understand the radian measure of the angle as a unit of measure. Note: Connected to F.TF.1 (+)

HSG.GPE.A.1 – • Derive the equation of a circle of given center and radius using the Pythagorean Theorem • Complete the square to find the center and radius of a circle given by an equation. Note: Students should also be able to identify the center and radius when given the equation of a circle and write the equation given the center and radius.

HSG.GPE.A.2 – (+) Derive the equation of a parabola given a focus and directrix.

HSG.GPE.A.3 – (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

HSG.GPE.B – Use coordinates to prove simple geometric theorems algebraically. [All standards in this cluster have to do with proof; a few are listed below]

HSG.GPE.B.4 – Use coordinates to prove simple geometric theorems algebraically. For example: Prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1,√3) lies on the circle centered at the origin and containing the point (0,2).

HSG.GPE.B.5 – • Prove the slope criteria for parallel and perpendicular lines. • Use the slope criteria for parallel and perpendicular lines to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

HSG.GMD.A.1 – Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. For example: Use dissection arguments, Cavalieri’s principle, and informal limit arguments.

HSG.GMD.A.3 – Cluster: Explain volume formulas and use them to solve problems. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems which may involve composite figures. • Compute the effect on volume of changing one or more dimension(s). For example: How is the volume affected by doubling, tripling, or halving a dimension?

7.EE.A – Use properties of operations to generate equivalent expressions.
1. Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.
2. Understand how the quantities in a problem are related by rewriting an expression in different forms. For example: \( a + 0.05a = 1.05a \) means that “increase by 5%” is the same as “multiply by 1.05.” Or the perimeter of a square with side lengths \( s \) can be written as \( s + s + s + s \) or \( 4s \).

8.EE.B.6 -- • Using a non-vertical or non-horizontal line, show why the slope \( m \) is the same between any two distinct points by creating similar triangles. • Write the equation \( y = mx \) for a line through the origin. • Be able to write the equation \( y = mx + b \) for a line intercepting the vertical axis at \( b \). [example in standards]
8.G.A.1 – **Verify experimentally the properties** of rotations, reflections, and translations: • Lines are taken to lines, and line segments to line segments of the same length. • Angles are taken to angles of the same measure. • Parallel lines are taken to parallel lines.

8.G.A.4 – **Understand** that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations.

• Given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

8.G.A.5 – **Use informal arguments to establish facts about**:

• The angle sum and exterior angle of triangles. *For example: Arrage three copies of the same triangle so that the sum of the three angles appears to form a line.*

• The angles created when parallel lines are cut by a transversal. *For example: Give an argument in terms of transversals about the angle relationships.*

• The angle-angle criterion for similarity of triangles.

8.G.B.6 – **Model or explain a proof of the Pythagorean Theorem and its converse.**
Conjectures at All Grade Levels

Which calculations will generate the same result as 121-89?

(a) 120-90  or  (b) 122-90

Why do you think so?
What do you notice?

- Choose an even and an odd integer. Multiply them together. What is the result?

- Compare your product to those in your group and record their results. What do you notice about all of the products? Does this always happen?

- Make a conjecture about the product of an even and an odd integer.

- Can you prove the conjecture?
Levels of Proof

Level 0 – No justification given
Level 1 – Appeal to external authority or rote procedures
Level 2 – Naïve reasoning, usually with incorrect conclusions.
Level 3A – Inductive reasoning with examples, experiments, or empirical demonstrations.
Level 3B – Inductive reasoning investigating if and why a generalization is held.
Level 4 – Transition to formal reasoning (elements of formal reasoning, but without the precision)
Level 5 – Formal reasoning

(Quinn, A.L, 2009)

Think about the conjecture, “The product of an even and an odd integer is an even integer.”

• What argument might a student functioning at each of these levels make?
• What question or task might help the student progress toward the next level or levels?

• Level 0

• Level 1

• Level 2

• Level 3a

• Level 3b

• Level 4

• Level 5
Even, Odd, and Beyond

Write a sample of student work you might see if a student is doing what is described in the standard. Consider how this would help them prepare for a proof that the product of an even and an odd integer is even integer.

- **2.OA.3** Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.

- **3.OA.9** Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.*

- **4.OA.5** Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *For example, given the rule “Add 3” and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.*

This work also prepares student to think about other number systems:

- **N.RN.3** Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.
Equations and Identities

1. Write down an example of an equation that has:
   (a) One solution.
   (b) Two solutions.
   (c) An infinite number of solutions.
   (d) No solutions.

2. For each of the following statements, indicate whether it is ‘Always true’, ‘Never true’ or ‘Sometimes true’. Circle the correct answer. If you choose ‘Sometimes true’ then state on the line below when it is true. The first one is done for you as an example.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Always true</th>
<th>Never true</th>
<th>Sometimes true</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + 2 = 3 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>It is true when ( x = 1 ).</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x - 12 = x + 30 )</td>
<td>Always true</td>
<td>Never true</td>
<td>Sometimes true</td>
</tr>
<tr>
<td>It is true when ( x = -12 ).</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 2(x + 6) = 2x + 12 )</td>
<td>Always true</td>
<td>Never true</td>
<td>Sometimes true</td>
</tr>
<tr>
<td>It is true when ( x = 0 ).</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 3(x - 2) = 3x - 2 )</td>
<td>Always true</td>
<td>Never true</td>
<td>Sometimes true</td>
</tr>
<tr>
<td>It is true when ( x = 0 ).</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( (x + 4)^2 = x^2 + 4x )</td>
<td>Always true</td>
<td>Never true</td>
<td>Sometimes true</td>
</tr>
<tr>
<td>It is true when ( x = -4 ).</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x^2 + 4 = 0 )</td>
<td>Always true</td>
<td>Never true</td>
<td>Sometimes true</td>
</tr>
<tr>
<td>It is true when ( x = \pm 2i ).</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Which of the equations in question 2 are also identities?

<table>
<thead>
<tr>
<th>Equation</th>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + 2 = 3 )</td>
<td></td>
</tr>
<tr>
<td>( x - 12 = x + 30 )</td>
<td></td>
</tr>
<tr>
<td>( 2(x + 6) = 2x + 12 )</td>
<td></td>
</tr>
<tr>
<td>( 3(x - 2) = 3x - 2 )</td>
<td></td>
</tr>
<tr>
<td>( (x + 4)^2 = x^2 + 4x )</td>
<td></td>
</tr>
<tr>
<td>( x^2 + 4 = 0 )</td>
<td></td>
</tr>
</tbody>
</table>

In your own words, explain what is meant by an identity.

<table>
<thead>
<tr>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
2-5b Student Work on “Sorting Equations and Identities”

**Video A** Notes:

Level of Proof (include evidence):

Next Instructional Step(s):

**Video B** Notes:

Level of Proof (include evidence):

Next Instructional Step(s):

**Video C** Notes:

Level of Proof (include evidence):

Next Instructional Step(s):
**Video D Notes:**

Level of Proof (include evidence):

Next Instructional Step(s):

**Video E Notes:**

Level of Proof (include evidence):

Next Instructional Step(s):
Asking Why: Different Ways to Find Slope

Describe **why** each of these methods can be used to find slope.

- “Rise over run”

- “Change in y over change in x”

- \[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

- The change in the range value, \( y \), when the domain value, \( x \), increases by 1
8.EE Slopes Between Points on a Line

Task

The slope between two points is calculated by finding the change in $y$-values and dividing by the change in $x$-values. For example, the slope between the points $(7, -15)$ and $(-8, 22)$ can be computed as follows:

- The difference in the $y$-values is $-15 - 22 = -37$.
- The difference in the $x$-values is $7 - (-8) = 15$.
- Dividing these two differences, we find that the slope is $\frac{-37}{15}$.

Eva, Carl, and Maria are computing the slope between pairs of points on the line shown below.

Eva finds the slope between the points $(0,0)$ and $(3,2)$. Carl finds the slope between the points $(3,2)$ and $(6,4)$. Maria finds the slope between the points $(3,2)$ and $(9,6)$. They have each drawn a triangle to help with their calculations (shown below).
i. Which student has drawn which triangle? Finish the slope calculation for each student. How can the differences in the \(x\)- and \(y\)-values be interpreted geometrically in the pictures they have drawn?

ii. Consider any two points \((x_1, y_1)\) and \((x_2, y_2)\) on the line shown above. Draw a triangle like the triangles drawn by Eva, Carl, and Maria. What is the slope between these two points? Why should this slope be the same as the slopes calculated by the three students?
Writing or Clarifying Definitions

- What is the definition of slope?

**Reflection:** Think about your thought process during this activity and discussion? What was hard about writing a definition of slope? ....what was easy?

How would the process of writing or clarifying definitions help your students think more deeply about content?
Proof that 2 = 1

Look through this proof and determine the assumptions that have been made (some true and some false). Which false assumption led to this incorrect conclusion?

Let $a$ and $b$ be real numbers such that $a = b$, then:

1) $a = b$

2) $a(a) = b(a)$

3) $a^2 = ab$

4) $a^2 - b^2 = ab - b^2$

5) $(a + b)(a - b) = b(a - b)$

6) $\frac{(a+b)(a-b)}{(a-b)} = \frac{b(a-b)}{(a-b)}$

7) $(a + b) = b$

8) $b + b = b$

9) $2b = b$

10) $\frac{2b}{b} = \frac{b}{b}$

11) $2 = 1$
More Attention to Assumptions

What “assumptions” (some true, some false) are made in the following proof that $\frac{x^2 - 1}{x - 1} = x + 1$?

Because $x^2 - 1 = (x - 1)(x + 1)$ then $\frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1}$. This can be written as $\frac{x-1}{x-1} \cdot \frac{x+1}{1}$.

Because any number divided by itself is 1, then

then $\frac{(x-1)(x+1)}{x-1} = \frac{x+1}{1}$. 
Why can’t we divide by zero? How would you explain this to students?

Showing the problem with division by zero using examples:

Proving division by zero is undefined:
Modes and Representations

<table>
<thead>
<tr>
<th>Modes</th>
<th>Representations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
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</tr>
</tbody>
</table>
Euclid’s *Elements*

Book I

**Definitions**

1. A *point* is that which has no part.
2. A *line* is breadthless length.
3. The extremities of a line are points.
4. A *straight line* is a line which lies evenly with the points on itself.
5. A *surface* is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A *plane surface* is a surface which lies evenly with the straight lines on itself.
8. A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called *rectilineal*.
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands.
11. An *obtuse angle* is an angle greater than a right angle.
12. An *acute angle* is an angle less than a right angle.
13. A *boundary* is that which is an extremity of anything.
14. A *figure* is that which is contained by any boundary or boundaries.
15. A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;
16. And the point is called the *centre* of the circle.
17. A *diameter* of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

Euclid’s definitions, postulates, and common notions—if Euclid is indeed their author—were not numbered, separated, or italicized until translators began to introduce that practice. The Greek text, however, as far back as the 1533 first printed edition, presented the definitions in a running narrative, more as a preface discussing how the terms would be used than as an axiomatic foundation for the propositions to come. We follow Heath’s formatting here. —Ed.
18. A **semicircle** is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle.

19. **Rectilineal figures** are those which are contained by straight lines, **trilateral** figures being those contained by three, **quadrilateral** those contained by four, and **multilateral** those contained by more than four straight lines.

20. Of trilateral figures, an **equilateral triangle** is that which has its three sides equal, an **isosceles triangle** that which has two of its sides alone equal, and a **scalene triangle** that which has its three sides unequal.

21. Further, of trilateral figures, a **right-angled triangle** is that which has a right angle, an **obtuse-angled triangle** that which has an obtuse angle, and an **acuteangled triangle** that which has its three angles acute.

22. Of quadrilateral figures, a **square** is that which is both equilateral and right-angled; an **oblong** that which is right-angled but not equilateral; a **rhombus** that which is equilateral but not right-angled; and a **rhomboid** that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called **trapezia**.

23. **Parallel** straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

**Postulates**

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

**Common Notions**

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

Distributive Property

In the time of Euclid, proofs of algebraic ideas were done geometrically. When he proved the distributive property, he used rectangles.

Can you use rectangles to make an argument that $3(x+4) = 3x + 12$?
BOOK II. PROPOSITIONS.

PROPOSITION I.

If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments.

Let $A, BC$ be two straight lines, and let $BC$ be cut at random at the points $D, E$; I say that the rectangle contained by $A, BC$ is equal to the rectangle contained by $A, BD$, that contained by $A, DE$ and that contained by $A, EC$.

For let $BF$ be drawn from $B$ at right angles to $BC$; let $BG$ be made equal to $A$ through $G$ let $GH$ be drawn parallel to $BC$, and through $D, E, C$ let $DK, EL, CH$ be drawn parallel to $BG$.

Then $BH$ is equal to $BK, DL, EH$.

Now $BH$ is the rectangle $A, BC$, for it is contained by $GB, BC$, and $BG$ is equal to $A$; $BK$ is the rectangle $A, BD$, for it is contained by $GB, BD$, and $BG$ is equal to $A$; and $DL$ is the rectangle $A, DE$, for $DK$, that is $BG$ is equal to $A$.

Similarly also $EH$ is the rectangle $A, EC$.

Therefore the rectangle $A, BC$ is equal to the rectangle $A, BD$, the rectangle $A, DE$ and the rectangle $A, EC$.

Therefore etc.

Q. E. D.
Growing Dots Pattern A

At the beginning  
At one minute  
At two minutes

Growing Dots Pattern B

At one minute  
At two minutes

Growing Dots Pattern C

At one minute  
At two minutes  
At three minutes
Can you make (and back up/support) a conjecture about any relationship that you see between these three growing patterns/functions. *Remember that conjectures must hold true for similar patterns beyond the problem set that we are examining.*
Conjecture Video

https://www.teachingchannel.org/videos/conjecture-lesson-plan

- How did the task and/or teacher questions and actions encourage students to think at a deeper level?

- What did you notice about the students (engagement, levels of thinking, misconceptions....)?

- When and how might you use this task or a similar task in instruction?

- If these were students in your class, what would be your next steps in instruction?
Patterns and Proof

Evaluate $n^2 + n + 41$ for integer values of $n$ from one through 10.

What do you notice?

Will what you noticed be true for all positive integers, $n$? Why or why not?
Difference in Squares Pattern

\[ 2^2 - 1^2 = 4 - 1 = 3 = 2 + 1 \]
\[ 3^2 - 2^2 = 9 - 4 = 5 = 3 + 2 \]
\[ 4^2 - 3^2 = 16 - 9 = 7 = 4 + 3 \]

- What patterns do you notice?

- Will the patterns always hold? Explain why or why not.
The Difference of Two Squares

Some integers can be written as the difference of two square integers.

For example:

\[ 9 \text{ can be written as } 5^2 - 4^2 \]

\[ 10^2 - 7^2 \]

Which other integers can be expressed as the difference of two square integers?
Write about any patterns you find and explain what you notice.

\[ 76 ? \]
\[ 100 ? \]
\[ 18 ? \]
\[ 623 ? \]
Survey of 123 Arkansas Student Responses to Question: “What is [mathematical] proof?”

115 students enrolled in Geometry, 28 enrolled in Algebra
(Survey Conducted Late September through October, 2014)

The italicized text in bullets represents sample student responses

<table>
<thead>
<tr>
<th>Only restated Question</th>
<th>Number of Students</th>
<th>Percent of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>A proof of math</td>
<td>39</td>
<td>27.3%</td>
</tr>
<tr>
<td>Something that can be proven by math.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>To Prove true</th>
<th>Number of Students</th>
<th>Percent of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>What makes or proves a statement is true</td>
<td>6</td>
<td>4.2%</td>
</tr>
<tr>
<td>When you prove something true</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>To Prove True or False</th>
<th>Number of Students</th>
<th>Percent of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>..when you prove something correct or incorrect.</td>
<td>5</td>
<td>3.5%</td>
</tr>
<tr>
<td>...when you prove something is true or not.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Evidence or Example</th>
<th>Number of Students</th>
<th>Percent of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive evidence... or Evidence...</td>
<td>20</td>
<td>14.0%</td>
</tr>
<tr>
<td>You’re showing someone something but giving them evidence to believe it.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A form of logical evidence.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Focus on Answer Only</th>
<th>Number of Students</th>
<th>Percent of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Using numbers/math [functions, equations...] to prove your answer is correct.</td>
<td>37</td>
<td>25.9%</td>
</tr>
<tr>
<td>An answer that can be backed by facts.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Something that gives you an idea of what your answer is.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Focus on Process and Answer</th>
<th>Number of Students</th>
<th>Percent of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proving a mathematical equation [function, expression, formula...] by working it out.</td>
<td>19</td>
<td>13.3%</td>
</tr>
<tr>
<td>Showing your work and explaining how you got the answer.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Make sure you are doing the right steps.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Figuring something out [solving] with math.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Focus on Process, Answer, and Reasons</th>
<th>Number of Students</th>
<th>Percent of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>To prove what the problem is and why it is.</td>
<td>2</td>
<td>1.4%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Other</th>
<th>Number of Students</th>
<th>Percent of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>A chart proving a mathematical statement (1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>If then statements. (2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hypothesis and conclusion (2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shows you have something (2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Says we need math in life (1)</td>
<td>13</td>
<td>9.1%</td>
</tr>
<tr>
<td>Math on paper (1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Proofing your work (1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>People that correct math (1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A formula (1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Something that will help you with math (1)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>No Answer</th>
<th>Number of Students</th>
<th>Percent of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>IDK or equivalent</td>
<td>2</td>
<td>1.4%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TOTAL</th>
<th>Number of Students</th>
<th>Percent of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>143</td>
<td>100.0%</td>
<td></td>
</tr>
</tbody>
</table>
What do you notice?

- Write down three consecutive integers and find their product. [Ex.: (3)(4)(5)=60]

- Compare your product to those in your group. What do you notice about all of the products?

- Make conjecture(s) about the product of three consecutive integers.
Criteria for a Valid Proof

What makes a valid proof? Write down criteria you believe make a valid proof.
Practicing Proof

**Algebra 1** – Find integer coordinates for each of the following equations. Make an argument that the coordinates you found are on the line:

a) \( y = 2x + 1 \)

b) \( y = 2x + \frac{1}{2} \)

c) \( 1620x + 1776y = 2014 \)

**Algebra 2** – Answer each of the questions and make an argument that your answers are true:

a) Is it possible to find a quadratic equation that has three solutions?

b) Is it possible to find an equation with two unknowns, \( x \) and \( y \), that has exactly one pair, \( (x, y) \), as a solution?

**Upper Level Math** – Is it possible to extend the inequality relation \(<\) from real numbers to complex numbers? Why or why not? In order to answer this question, you need to know the following fundamental properties of the inequality relation:

a) Any two distinct numbers can be compared; one of them is greater than the other (if \( a \neq b \), then either \( a < b \) or \( b < a \)).

b) The product of two positive numbers is positive (if \( 0 < a \) and \( 0 < b \) then \( 0 < ab \)).

c) The product of two negative numbers is positive (if \( a < 0 \) and \( b < 0 \) then \( 0 < ab \)).

A-CED Products and Reciprocals

Task

The product of two positive numbers is 9. The reciprocal of one of these numbers is 4 times the reciprocal of the other number. What is the sum of the two numbers?
A-REI Reasoning with linear inequalities

Task

The following is a student solution to the inequality

\[
\frac{5}{18} - \frac{x - 2}{9} \leq \frac{x - 4}{6}.
\]

\[
\frac{5}{18} - \frac{2x - 2}{9} \leq \frac{3x - 4}{3}.
\]

\[
\frac{5}{18} - \frac{2x - 2}{18} \leq \frac{3x - 4}{18}.
\]

\[
5 - (2x - 2) \leq 3x - 4.
\]

\[
5 - 2x + 2 \leq 3x - 4.
\]

\[
7 - 2x \leq 3x - 4.
\]

\[
-5x \leq -11.
\]

\[
x \leq \frac{11}{5}.
\]

a. There are two mathematical errors in this work. Identify at what step each mathematical error occurred and explain why they are mathematically incorrect.

The first mathematical error occurred going from line ____ to line ____.

Why it is incorrect:
The second mathematical error occurred going from line ___ to line ____.

Why it is incorrect:

b. How would you help the student understand his mistakes?

c. Solve the inequality correctly.
A-REI An Extraneous Solution

Task

Megan is working solving the equation

\[ \frac{2}{x^2 - 1} - \frac{1}{x - 1} = \frac{1}{x + 1}. \]

She says

*If I clear the denominators I find that the only solution is \( x = 1 \) but when I substitute in \( x = \) the equation does not make any sense.*

a. Is Megan's work correct?

b. Why does Megan's method produce an \( x \) value that does not solve the equation?
Proving Truth or Falsehood...

Which of these are true statements? How do you know?

a) \(1=1\)

b) \(0.\bar{3} = \frac{1}{3}\)

c) \(0.\bar{9} = 1\)
Quadratic Formula

Solve $3x^2 + 4x - 5 = 0$, by completing the square.

Prove that in a quadratic equation of the form, $ax^2 + bx + c = 0$, the solutions (real or complex) are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
Engaging Students in Proof and Reasoning in HS non-Geometry Courses (TESS 3d)

References


Additional Resources

http://hardingmathspecialist.pbworks.com – Links to many helpful resources. Lessons and tasks are on the “Math Resources” link. If interested in engaging contextual problems that start with pictures or videos, see the section labeled “Three-Act Lessons.” There is also a page called “High Probability (High Yield) Strategies Resources” that contains many strategies that have been shown to increase student learning and student engagement.

More Proof Problems

**Algebra 1** – Problem 1: Under a dilation of the plane with scale factor, \( r \), each coordinate point \((x, y)\) is mapped to \((rx, ry)\). Use this fact to prove that a dilation of scale factor, \( r \), maps a line to a parallel line.

Problem 2: What value makes \(4x - 1 = 3\) true? Is that the only value? Justify your answer.

**Algebra 2** – When are the equations \(\frac{(x+1)(x+2)}{(x+1)} = 5\) and \(x + 2 = 5\) equivalent? Justify your answer.

**Algebra 2 and Upper Level Math** –

1. Perform the indicated operations: \((x - 1)(x + 1); \ (x - 1)(x^2 + x + 1)\).
2. Without doing any algebraic manipulation, anticipate the result of the following product: \((x - 1)(x^3 + x^2 + x + 1)\).
3. Verify the above result using paper and pencil and then using the calculator.
4. What do the following three expressions have in common? And, also, how do they differ? \((x - 1)(x + 1); \ (x - 1)(x^2 + x + 1); \ (x - 1)(x^3 + x^2 + x + 1)\).
5. How do you explain the fact that when you multiply: i) the two binomials above, ii) the binomial with the trinomial above, and iii) the binomial with the quadrinomial above, you always obtain a binomial as the product?
6. On the basis of the expressions we have found so far, predict a factorization of the expression \(x^5 - 1\).
7. Explain why the product \((x - 1)(x^{15} + x^{14} + x^{13} + \cdots + x^2 + x + 1)\) gives the result \(x^{16} - 1\).

Conjecture, in general, for what numbers, \(n\) will the factorization of \(x^n - 1\):

i. contain exactly two factors?
ii. contain more than two factors?
iii. include \((x + 1)\) as a factor?

Can you prove your conjecture(s)?

Upper level math problem adapted from:

Proof non-Geometry
Day 1: 3 - 2 - 1 Reflection

3 things I found interesting

2 things I found worthwhile

1 thing I found challenging